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On Preserving Structured Matrices Using Double Bracket Operators: Tridiagonal and Toeplitz Matrices

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ON PRESERVING STRUCTURED MATRICES USING DOUBLE BRACKET OPERATORS: TRIDIAGONAL AND TOEPLITZ MATRICES

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Abstract

In the algebra of square matrices over the complex numbers, $[X, Y]$ denotes $XY - YX$. Two problems are solved: (1) Find all Hermitian matrices M which have the following property: For every Hermitian matrix A, if A is tridiagonal, then so is $[A, [A, M]]$. (2) Find all Hermitian matrices M which have the following property: For every Hermitian matrix A, if A is Toeplitz, then so is $[A, [A, M]]$.

1. Introduction

In this paper all matrices are square and complex and the set of all

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$n \times n$ complex matrices is denoted by $\mathbf{C}^{n \times n}$. We use the Lie bracket notation, $[X, Y] = XY - YX$. We call the map $X \to [X, [X, M]]$ on $\mathbf{C}^{n \times n}$ the *double bracket* map determined by M. Note that if A and M are Hermitian, then $[A, M]$ is skew-Hermitian and $[A, [A, M]]$ is Hermitian. In other words, if M is Hermitian, then the associated double bracket map is "Hermitian preserving." The following two problems are considered:

Problem A. Find all Hermitian matrices M with the "Jacobi preserving" property: For every Hermitian matrix A, if A is tridiagonal, then so is $[A, [A, M]]$.

Problem B. Find all Hermitian matrices M with the "Toeplitz preserving" property: For every Hermitian matrix A, if A is Toeplitz, then so is $[A, [A, M]]$.

We shall present the solutions, using elementary techniques only, in Sections 2 and 3 respectively.

Before presenting our solutions we want to explain why we chose to study these problems. The need to calculate the eigenvalues of large structured matrices leads to them. Consider the initial value problem with the following "double bracket" ordinary differential equation:

$$X' = [X, [X, M]], \qquad X(0) = A,$$

where M and A are Hermitian matrices. The solution $X(t)$ of this problem has very interesting properties summarized in Proposition 1.2. These properties are known in literature. Chu and Driessel [9] considered the optimization problem: Given square Hermitian matrices A and M, find the matrix which is closest to M subject to the constraint that it is unitarily similar to A. They derived the (gradient) double bracket differential equation from this optimization problem. Bloch, Brockett and Ratiu [2] studied this differential equation and showed that if M is the diagonal matrix with diagonal entries $1, 2, \ldots, n$, then the double bracket differential equation is closely related to the Toda flow and the QR algorithm. See Nanda [20], Brockett [3, 4, 5], Bloch [1], and Chu [11]. The connection between the Toda flow and the QR algorithm is due to Symes [22, 23]. Deift, Nanda and Tomei [12] emphasized the use of isospectral flows to find eigenvalues. Chu [6] and Watkins [27] wrote survey articles concerning these matters. For description of the QR algorithm and its origins see, for example, Parlett [21] or Golub and Van Loan [19]. There are differential equations related to the Jacobi methods for finding eigenvalues; see Driessel [14, 15]. For descriptions of traditional Jacobi methods, see, for example, Parlett [21], Golub and Van Loan [19] or Wilf [28]. For more on the differential geometry of isospectral surfaces, see Tomei [25] and Driessel [17]. There are also differential equations for finding singular values; see Chu [7], Driessel [16], Deift, Demmel, Li and Tomei [13].

The following proposition shows that if a smooth vector field preserves a linear structure, then so does the corresponding flow. This result is an instance of more general results concerning vector fields tangent to manifolds. See, for example, Thorpe [24], Chapter 5 "Vector fields on surfaces; orientation."

Proposition 1.1. *Let V be a normed linear space over the real numbers and let U be a subspace of V. Let $F : V \to V$ be a continuously differentiable function satisfying $F(U) \subseteq U$. Let a be an element of U and let $x(t)$ be the solution of the initial value problem: $x' = F(x)$, $x(0) = a$. Then the solution curve $x(t)$ lies in U.*

Proof. Let W be a subspace of V which is a complement of U; in symbols, $V = U \oplus W$. Let P_W and P_U be the linear projections of V onto W and U determined by this direct sum decomposition. Consider the initial value problem: $y' = F(P_U(y))$, $y(0) = a$. Let $y(t)$ denote the solution of this initial value problem. Note that $[P_W(y)]' = P_W(y') = P_W(F(P_U(y))) = 0$. Hence $P_W(y(t))$ is a constant. Since $P_W(y(0)) = P_W(a) = 0$, this constant is 0. Thus $y(t)$ is in U. It follows that $y(t)$ is a solution of the initial value problem that determines $x(t)$. By the uniqueness of the solution of this latter problem, we have $x(t) = y(t)$ and $x(t)$ is in U.

The next proposition shows that we can use the double bracket differential equation to find the eigenvalues of Hermitian matrices.

Proposition 1.2. *Let A and M be Hermitian matrices and let $X(t)$ be the solution of the following initial value problem* (*):

$$X' = [X, [X, M]], \qquad X(0) = A.$$

Then this solution has the following properties:

(1) *For all t, $X(t)$ is a Hermitian matrix with the same eigenvalues as A.*

(2) *The solution $X(t)$ converges to a Hermitian matrix which commutes with M.*

(3) *If M is a diagonal matrix with distinct diagonal entries, then the solution $X(t)$ converges to a diagonal matrix with diagonal entries equal to the eigenvalues of A.*

Proof. It follows from Proposition 1.1 that the solution $X(t)$ is Hermitian for all t since double bracket map is Hermitian preserving. Define $K(t) = [X(t), M]$. Note $K(t)$ is skew Hermitian since $X(t)$ and M are Hermitian. We consider the following initial value problem (**):

$$Q' = QK, \qquad Q(0) = I.$$

Claim. The solution $Q(t)$ of (**) is unitary for all t.

Note that $(QQ^*)' = Q'Q^* + Q(Q')^* = QKQ^* - QKQ^* = 0$ since K is skew Hermitian. Hence $Q(t)Q^*(t)$ is constant. This constant is I since $Q(0)Q(0)^* = I$.

Claim. The solution $X(t)$ of (*) satisfies $X(t) = Q(t)^* AQ(t)$ for all t.

Define $Y(t) := Q(t)^* AQ(t)$. Then

$$Y' = (Q^*AQ)' = (Q')^* AQ + Q^*AQ' = -KQ^*AQ + Q^*AQK = [Y, K].$$

Thus, Y is a solution of the following initial value problem:

$$Y' = [Y, K], \quad Y(0) = A.$$

But X is also a solution of this initial value problem. Hence by uniqueness of such solutions, $X = Y$. Hence (1) is proved.

We shall regard $\mathbf{C}^{n \times n}$ as a real vector space (with dimension $2n^2$) with the real inner product:

$$(P, Q) := Re(Trace(PQ^*)),$$

where $Re\, z$ denotes the real part of the complex number z and $Trace(R)$ denotes the trace of the matrix R. In terms of coordinates $(P, Q) = \sum_{i,j} (Re(P_{ij})\, Re(Q_{ij}) + Im(P_{ij})\, Im(Q_{ij}))$, where $Im\, z$ denotes the imaginary part of a complex number z. We have

$$\frac{d}{dt}\left(\frac{1}{2} \| X(t) - M \|^2\right) = -([X, M], [X, M]) \leq 0 \qquad (\dagger)$$

since

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}(X - M, X - M) &= (X - M, X') \\ &= (X - M, [X, [X, M]]) \\ &= ([X, X - M], [X, M]) \\ &= -([X, M], [X, M]). \end{aligned}$$

We use the fact that $(PQ, R) = (Q, P^*R) = (P, RQ^*)$ and $([P, Q], R) = (Q, [P^*, R])$. We see that the solution $X(t)$ remains bounded since $\| X(t) - M \| \leq \| A - M \|$. We also see from (†) that $X(t)$ converges to a matrix Y which satisfies $[M, Y] = 0$. Thus, $X(t)$ satisfies property (2).

Now assume that $M = Diag(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with distinct diagonal entries. Then $[M, Y]_{ij} = (d_i - d_j) Y_{ij}$. Hence $[M, Y] = 0$

implies that Y is a diagonal matrix. Thus we see that property (3) follows from properties (1) and (2).

There are computer softwares available for numerically solving differential equations $X' = F(X)$. The evaluations of the function F usually constitute most of the numerical work. For the double bracket differential equation we have $F(X) = [X, [X, M]]$. Generally the evaluation of the double bracket requires $O(n^3)$ operations. We can reduce this work if the double bracket operation preserves structure. For example, if X are tridiagonal, then the bracket computation only requires $O(n^2)$ operations. Consequently, we are interested in structure preserving properties of the double bracket.

Actually we do not predict that isospectral flows will be directly used to effectively find eigenvalues. However, we do hope that the study of such flows will lead to matrix factorizations which do so. For more on the relationship between isospectral flows and matrix factorizations see Chu [6], Watkins [27] and. Chu and Norris [11].

We are particularly interested in operators on matrices that preserve both spectrum and Toeplitz or near Toeplitz structure. There has been some study of such linear operators. Chu [10] obtained the following result. We use J to denote the $n \times n$ exchange matrix $[J_{ij}]$, where $J_{ij} = 1$ if $i + j = n + 1$ otherwise $J_{ij} = 0$, and I' to denote the matrix $Diag(-1, +1, -1, \ldots, (-1)^{n-1})$.

Proposition 1.3. *Let Q be a real orthogonal matrix. Then the following conditions are equivalent:*

(1) *For every real symmetric matrix A, if A is Toeplitz, then so is QAQ^t.*

(2) *The matrix Q is one of the following:* $\pm I$, $\pm J$, $\pm I'$, $\pm I'J$.

Driessel and So [18] obtained the following related result. For complex number λ, we use $D(\lambda)$ to denote $Diag(1, \lambda, \ldots, \lambda^{n-1})$.

Proposition 1.4. *Let T be a nonzero linear map on the real space of Hermitian matrices. Then the following conditions are equivalent:*

(1) *For every Hermitian matrix A, the spectrum of A is the same as the spectrum of $T(A)$ and the displacement rank of A is the same as the displacement rank of $T(A)$.*

(2) *There is a complex number λ such that $|\lambda| = 1$ and either for all Hermitian A, $T(A) = D(\lambda)AD(\lambda)^*$ or for all Hermitian A, $T(A) = D(\lambda)A^t D(\lambda)^*$.*

These results show that there are very few linear operators preserving spectrum and Toeplitz structure. Consequently, we begin to study double operators that do so. Unfortunately, we shall find that there are also very few double bracket operators that do so.

2. Preserving Hermitian Tridiagonal Matrices

We shall say that $A \in \mathbf{C}^{n \times n}$ is a *Jacobi* matrix if A is Hermitian and tridiagonal. In this section we consider the following:

Problem A. Find all Hermitian matrices M with the "Jacobi preserving" property (J):

if A is Jacobi, then so is $[A, [A, M]]$.

Let $\mathcal{J}(n)$ denote the set of Hermitian matrices which satisfy the Jacobi preserving property; in symbols,

$$\mathcal{J}(n) := \{M \in \mathbf{C}^{n \times n} : M^* = M \text{ and } M \text{ satisfies } (J)\}.$$

For A, let $ad(A)$ denote the linear map $X \to [A, X]$ on $\mathbf{C}^{n \times n}$. In the theory of Lie algebras this map is called the *adjoint* map determined by A. Let $Jac(n)$ denote the set of Jacobi matrices with size n. Note that $Jac(n)$ is a real linear subspace of $\mathbf{C}^{n \times n}$ and hence that $ad(A)^{-1}(Jac(n)) = \{X \in \mathbf{C}^{n \times n} : [A, X] \in Jac(n)\}$ is also a real linear subspace. Note further that a Hermitian matrix M satisfies property (J) iff for every $A \in Jac(n)$, $M \in (ad(A) \circ ad(A))^{-1}(Jac(n))$; in symbols,

$$\mathcal{J}(n) = \bigcap \{(ad(A) \circ ad(A))^{-1}(Jac(n)) : A \in Jac(n)\} \bigcap Herm(n),$$

where $Herm(n)$ is the set of all $n \times n$ Hermitian matrices. Thus we see that $\mathcal{J}(n)$ is the intersection of certain real subspaces of $\mathbf{C}^{n \times n}$ and, in particular, we see that $\mathcal{J}(n)$ is itself such a subspace. To solve the given problem we shall adopt the following strategy: We select $A \in Jac(n)$ for which it is easy to compute $ad(A)$. We then determine $(ad(A) \circ ad(A))^{-1}(Jac(n))$. In this way we reduce the possible forms for $M \in \mathcal{J}(n)$.

The simplest choice for A is a diagonal matrix. Let D be a diagonal matrix with real distinct diagonal entries $d_1, d_2, \ldots, d_n$; in symbols, $D = Diag(d_1, d_2, \ldots, d_n)$. Then $[D, X](i, j) = (d_i - d_j)X(i, j)$ and $[D, [D, X]](i, j) = (d_i - d_j)^2 X(i, j)$. Thus, we see that if $[D, [D, M]] \in Jac(n)$, then M must be a Jacobi matrix. In other words, M must have the following form:

$$M = \begin{bmatrix} x_1 & y_1 & 0 & \cdots & 0 & 0 \\ \overline{y_1} & x_2 & y_2 & \cdots & 0 & 0 \\ 0 & \overline{y_2} & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1} & y_{n-1} \\ 0 & 0 & 0 & \cdots & \overline{y_{n-1}} & x_n \end{bmatrix}.$$

We can easily find nondiagonal test matrices which convince us that M must be a diagonal matrix. In fact, let E_{pq} denote the matrix with 1 in the (p, q) position and 0's elsewhere; in symbols, $E_{pq}(i, j) := \delta(p, i)\delta(q, j)$, where δ is the Kronecker delta function. Take $A := E_{11} + E_{23} + E_{32}$. Computing with 3×3 matrices shows what happens. We have (using $*$ for "don't care" positions)

$$[A, M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 0 \\ \bar{y}_1 & x_2 & y_2 \\ 0 & \bar{y}_2 & x_3 \end{bmatrix} - \begin{bmatrix} x_1 & y_1 & 0 \\ \bar{y}_1 & x_2 & y_2 \\ 0 & \bar{y}_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} * & y_1 & 0 \\ 0 & * & * \\ \bar{y}_1 & * & * \end{bmatrix} - \begin{bmatrix} * & 0 & y_1 \\ \bar{y}_1 & * & * \\ 0 & * & * \end{bmatrix} = \begin{bmatrix} * & y_1 & -y_1 \\ -\bar{y}_1 & * & * \\ \bar{y}_1 & * & * \end{bmatrix}$$

and

$$[A, [A, M]] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} * & y_1 & -y_1 \\ -\bar{y}_1 & * & * \\ \bar{y}_1 & * & * \end{bmatrix} - \begin{bmatrix} * & y_1 & -y_1 \\ -\bar{y}_1 & * & * \\ \bar{y}_1 & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} * & * & -y_1 \\ * & * & * \\ -\bar{y}_1 & * & * \end{bmatrix} - \begin{bmatrix} * & * & y_1 \\ * & * & * \\ \bar{y}_1 & * & * \end{bmatrix} = \begin{bmatrix} * & * & -2y_1 \\ * & * & * \\ -2\bar{y}_1 & * & * \end{bmatrix}.$$

We see that y_1 must be zero since $[A, [A, M]]$ is Jacobi. Now, take $A = E_{12} + E_{21} + E_{33}$, then we have

$$[A, M] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 0 \\ \bar{y}_1 & x_2 & y_2 \\ 0 & \bar{y}_2 & x_3 \end{bmatrix} - \begin{bmatrix} x_1 & y_1 & 0 \\ \bar{y}_1 & x_2 & y_2 \\ 0 & \bar{y}_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} * & * & y_2 \\ * & * & 0 \\ 0 & \bar{y}_2 & * \end{bmatrix} - \begin{bmatrix} * & * & 0 \\ * & * & y_2 \\ \bar{y}_2 & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & y_2 \\ * & * & -y_2 \\ -\bar{y}_2 & \bar{y}_2 & * \end{bmatrix}$$

and

$$[A, [A, M]] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} * & * & y_2 \\ * & * & -y_2 \\ -\bar{y}_2 & \bar{y}_2 & * \end{bmatrix} - \begin{bmatrix} * & * & y_2 \\ * & * & -y_2 \\ -\bar{y}_2 & \bar{y}_2 & * \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} * & * & -y_2 \\ * & * & * \\ -\overline{y}_2 & * & * \end{bmatrix} - \begin{bmatrix} * & * & y_2 \\ * & * & * \\ \overline{y}_2 & * & * \end{bmatrix} = \begin{bmatrix} * & * & -2y_2 \\ * & * & * \\ -2\overline{y}_2 & * & * \end{bmatrix}.$$

We see that y_2 must be zero since $[A, [A, M]]$ is Jacobi. Similarly, we see that the rest of the off-diagonal elements of M must be zero; in symbols, $y_1 = y_2 = \dots = y_{n-1} = 0$. In other words, $M = Diag(x_1, x_2, \dots, x_n)$.

Now, let Z be the matrix with ones on the first superdiagonal and zeros elsewhere; in symbols, $Z = E_{12} + E_{23} + \dots + E_{n-1,\, n}$. We take $A = Z + Z^t$. Note that for any matrix W,

$$[A, W](i, j) = W(i-1, j) + W(i+1, j) - W(i, j-1) - W(i, j+1).$$

We have used the convention that $W(i, j) = 0$ if either i or j is not between 1 and n. Computing with 4×4 matrices we see what happens. We have

$$[A, M] = \begin{bmatrix} 0 & x_2 - x_1 & 0 & 0 \\ x_1 - x_2 & 0 & x_3 - x_2 & 0 \\ 0 & x_2 - x_3 & 0 & x_4 - x_3 \\ 0 & 0 & x_3 - x_4 & 0 \end{bmatrix}$$

and

$$[A, [A, M]] = \begin{bmatrix} * & * & x_3 - 2x_2 + x_1 & 0 \\ * & * & * & x_4 - 2x_3 + x_2 \\ x_3 - 2x_2 + x_1 & * & * & * \\ 0 & x_4 - 2x_3 + x_2 & * & * \end{bmatrix}.$$

Thus, in general, we see that the second differences $x_{k+1} - 2x_k + x_{k-1}$ for $k = 2, 3, \dots, n-1$ must be zero. It follows that there exist real numbers a and b such that $x_k = a + bk$ for $k = 1, 2, \dots, n$. In other words, $M = aI + bD$, where $D := Diag(1, 2, \dots, n)$. We have obtained the solution of Problem A.

Theorem 2.1. *Let $n \geq 3$ and let M be a Hermitian matrix. Then the double bracket operator determined by M has the Jacobi preserving property iff M is a real linear combination of I and $Diag(1, 2, \ldots, n)$.*

Proof. It is easy to verify that I and D have the Jacobi preserving property. On the other hand, if M has the Jacobi preserving property, then M must have the given form according to the argument preceding the statement of the theorem.

3. Preserving Hermitian Toeplitz Matrices

Recall that a matrix $A \in \mathbf{C}^{n \times n}$ is "Toeplitz" if it has constant diagonals; in symbols, $A(i, j) = A(i+1, j+1)$ for all $i, j = 1, 2, \ldots, n-1$. In this section, we consider the following:

Problem B. Find all Hermitian matrices M with the "Toeplitz preserving" property (T):

if A is Toeplitz and Hermitian, then so is $[A, [A, M]]$.

Let $\mathcal{T}(n)$ denote the set of Hermitian matrices which satisfy the Toeplitz preserving property. Let $Toep(n)$ denote the set of Hermitian Toeplitz matrices with size n. Note that $Toep(n)$ is a real linear subspace of $\mathbf{C}^{n \times n}$. Also, note that $M \in \mathcal{T}(n)$ iff for every $A \in Toep(n)$, $M \in (ad(A) \circ ad(A))^{-1}(Toep(n))$; in symbols,

$$\mathcal{T}(n) = \bigcap \{(ad(A) \circ ad(A))^{-1}(Toep(n)) : A \in Toep(n)\}.$$

As in the previous section, we will use test matrices from $Toep(n)$ to reduce the possible forms for M. We first turn to the real version of Problem B. We shall see that the solution of this problem will provide part of the solution of Problem B.

Lemma 3.1. *Let M be a real symmetric matrix, then the following properties for M are equivalent:*

(1) *if A is real symmetric and Toeplitz, then so is $[A, [A, M]]$;*

(2) *if* A *and* B *are real symmetric and Toeplitz, then so is* $[A, [B, M]] + [B, [A, M]]$;

(3) *if* $A_1, \ldots, A_m$ *form a basis for real symmetric matrices, then* $[A_i, [A_j, M]] + [A_j, [A_i, M]]$ *is real symmetric and Toeplitz.*

Proof. (1) $\Rightarrow$ (2) follows from the identity:

$$[A + B, [A + B, M]] = [A, [A, M]] + [A, [B, M]] + [B, [A, M]] + [B, [B, M]].$$

(2) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (1), consider a real symmetric matrix A. Since $\{A_i\}$ is a basis, $A = \sum_i a_i A_i$. Hence

$$[A, [A, M]] = \sum_{i,\, j} a_i a_j ([A_i, [A_j, M]] + [A_j, [A_i, M]])$$

is also real symmetric.

Again we let Z be the matrix with ones on the first superdiagonal and zeros elsewhere. Let $H_p = Z^p + (Z^p)^t$. Then the set $\{I\} \cup \{H_p : p = 1, 2, \ldots, n - 1\}$ is a basis for real symmetric matrices. Recall that the "exchange" matrix J and note that $JAJ = A$ for every real symmetric Toeplitz matrix A, hence $[A, J] = 0$.

Theorem 3.2. *Let M be a real symmetric matrix. Then M satisfies the real Toeplitz preserving property (TR):*

if A is real symmetric Toeplitz, then so is $[A, [A, M]]$

if and only if M is a real linear combination of I and J.

Proof. ($\Leftarrow$) Let $M = aI + bJ$, where a and b are real numbers. Then $[A, M] = 0$ for every symmetric Toeplitz matrix A.

($\Rightarrow$) **Claim 1.** If $[H_1, K]$ is Toeplitz and K is real skew-symmetric, then $K = 0$.

We shall outline the proof of this claim for 5×5 matrices. It will be clear that the argument applies to the general case. Assume $[H_1, K]$ is Toeplitz and that K is skew-symmetric. Then K has the following form:

$$K = \begin{bmatrix} 0 & x_1 & * & * & * \\ -x_1 & 0 & x_2 & * & * \\ * & -x_2 & 0 & x_3 & * \\ * & * & -x_3 & 0 & x_4 \\ * & * & * & -x_4 & 0 \end{bmatrix}.$$

We do not care about the $*$ entries until later. Since $[H_1, K]$ is Toeplitz and the trace of $[H_1, K]$ is 0 we see that the diagonal entries are zero. In other words we have $0 = -2x_1 = 2(x_1 - x_2) = 2(x_2 - x_3) = 2(x_3 - x_4) = 2x_4$. We conclude that $0 = x_1 = x_2 = x_3 = x_4$. Thus, K has the following form:

$$K = \begin{bmatrix} 0 & 0 & y_1 & * & * \\ 0 & 0 & 0 & y_2 & * \\ -y_1 & 0 & 0 & 0 & y_3 \\ * & -y_2 & 0 & 0 & 0 \\ * & * & -y_3 & 0 & 0 \end{bmatrix}.$$

We consider the entries of $[H_1, K]$ which lie on the first superdiagonal. We have $[H_1, K]_{i,\, i+1} = a$ for some real scalar a and we also have, by direct computation

i	$[H_1, K]_{i,\, i+1}$
1	$-y_1$
2	$y_1 - y_2$
3	$y_2 - y_3$
4	y_3

Adding we get $4a = \sum_{i=1}^{4} [H_1, K]_{i,\, i+1} = 0$. We conclude that $a = 0$ and so $y_1 = y_2 = y_3 = 0$. Clearly we can continue in this way showing that successive superdiagonals of K are zero.

Claim 2. Let M be a real symmetric matrix which satisfies the property (TR). Then $[H_p, M] = 0$ for $p = 1, 2, \ldots, n-1$.

Since H_1 is real symmetric Toeplitz, $[H_1, [H_1, M]]$ is Toeplitz. Then, by Claim 1, $[H_1, M] = 0$ since $[H_1, M]$ is real skew-symmetric. Note that $\{I\} \cup \{H_p : p = 1, \ldots, n-1\}$ forms a basis for real symmetric Toeplitz. Hence, by Lemma 3.1, $[H_1, [H_p, M]] + [H_p, [H_1, M]] = [H_1, [H_p, M]]$ is Toeplitz since M has (TR) property. Consequently, by Claim 1, $[H_p, M] = 0$ since $[H_p, M]$ is real skew-symmetric.

Claim 3. If M commutes with every real symmetric Toeplitz matrix, then M is a real linear combination of I and J.

Assume M commutes with every symmetric Toeplitz matrix. Decompose M as follows:

$$M = \begin{bmatrix} a & x^t & b \\ x & P & y \\ b & y^t & c \end{bmatrix},$$

where a, b and c are real scalars, x and y are real vectors with length $n-2$, and P is an $(n-2) \times (n-2)$ real symmetric matrix. By Claim 2, we have

$$\begin{aligned} 0 &= [H_{n-1}, M] \\ &= \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a & x^t & b \\ x & P & y \\ b & y^t & c \end{bmatrix} - \begin{bmatrix} a & x^t & b \\ x & P & y \\ b & y^t & c \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} b & y^t & c \\ 0 & 0 & 0 \\ a & x^t & b \end{bmatrix} - \begin{bmatrix} b & 0 & a \\ y & 0 & x \\ c & 0 & b \end{bmatrix} = \begin{bmatrix} 0 & y^t & c-a \\ -y & 0 & -x \\ a-c & x^t & 0 \end{bmatrix}. \end{aligned}$$

We conclude that $a = c$ and $x = y = 0$. In other words, M has the following form

$$M = \begin{bmatrix} a & 0 & b \\ 0 & P & 0 \\ b & 0 & a \end{bmatrix}.$$

Let $N := M - aI - bJ$. Then N has the form

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where R is an $(n-2) \times (n-2)$ matrix. Note that N commutes with every symmetric Toeplitz matrix. We want to see that $R = 0$. Let A be an arbitrary symmetric Toeplitz matrix. Decompose A as follows:

$$A = \begin{bmatrix} p & u^t & q \\ u & T & Ju \\ q & (Ju)^t & p \end{bmatrix},$$

where p and q are scalars, u is a vector with length $n-2$ and T is the $(n-2) \times (n-2)$ symmetric Toeplitz matrix determined by p and u. We consider $[A, N]$. We have

$$0 = \begin{bmatrix} p & u^t & q \\ u & T & Ju \\ q & (Ju)^t & p \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p & u^t & q \\ u & T & Ju \\ q & (Ju)^t & p \end{bmatrix}$$

$$= \begin{bmatrix} * & * & * \\ Ru & * & * \\ * & * & * \end{bmatrix}.$$

From this we see that $Ru = 0$ for every vector u. Thus $R = 0$.

Lemma 3.3. *If the last three rows and columns of X are zero and $[H_1, [H_1, X]]$ is Toeplitz, then $X = 0$.*

Proof. We shall outline the proof of the claim for 6×6 matrices. It will be clear that the argument applies to the general case. Let

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have

$$[H_1, X] = \begin{bmatrix} * & * & * & -x_{13} & 0 & 0 \\ * & * & * & -x_{23} & 0 & 0 \\ * & * & * & -x_{33} & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[H_1, [H_1, X]] = \begin{bmatrix} * & * & * & * & x_{13} & 0 \\ * & * & * & * & x_{23} & 0 \\ * & * & * & * & x_{33} & 0 \\ * & * & * & * & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix is Toeplitz we conclude that $x_{13} = x_{23} = x_{33} = 0$ and $x_{31} = x_{32} = x_{33} = 0$. In other words, the last four rows and columns of X are zero. Clearly we can continue showing more rows and columns are zero. Finally, we get $X = 0$.

Theorem 3.4. *Let N be a real skew-symmetric matrix with dimension greater than 3. If N has the property (TS):*

$$\text{if } A \text{ is Hermitian Toeplitz, then so is } [A, [A, N]],$$

then $N = 0$.

Proof. Assume that N is a real skew-symmetric matrix which satisfies the property (*TS*). Again, we use appropriate test matrices to discover the form of N. We first use matrices H_p, where $p > n/2$. Note that such a matrix has the following block form:

$$H_p = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix},$$

where $I = I_{n-p}$ is the $(n-p) \times (n-p)$ identity matrix. For any matrix X we partition as follows:

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}.$$

Then

$$[H_p, [H_p, X]] = \begin{bmatrix} 2(X_{11} - X_{33}) & X_{12} & 2(X_{13} - X_{31}) \\ X_{21} & 0 & X_{23} \\ 2(X_{31} - X_{13}) & X_{32} & 2(X_{33} - X_{11}) \end{bmatrix}.$$

In particular, when $p = n - 1$ we partition N as follows:

$$N = \begin{bmatrix} 0 & x^t & b \\ -x & N_1 & y \\ -b & -y^t & 0 \end{bmatrix},$$

where b is a scalar, x and y are vectors and N_1 is an $(n-2) \times (n-2)$ real skew-symmetric matrix. We get

$$[H_{n-1}, [H_{n-1}, N]] = \begin{bmatrix} 0 & x^t & 4b \\ -x & 0 & y \\ -4b & -y^t & 0 \end{bmatrix}.$$

This matrix must be Toeplitz. We conclude that there is a scalar a such that

$$x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a \end{bmatrix}, \qquad y = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

In other words, N has the following form:

$$N = \begin{bmatrix} 0 & 0 & \cdots & a & b \\ 0 & & & & a \\ \vdots & & N_1 & & \vdots \\ -a & & & & 0 \\ -b & -a & \cdots & 0 & 0 \end{bmatrix}.$$

When $p = n - 2$, we partition N as follows:

$$N = \begin{bmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & x^t & b' & a \\ 0 & -x & N_2 & y & 0 \\ -a & -b' & -y^t & 0 & 0 \\ -b & -a & 0 & 0 & 0 \end{bmatrix}.$$

We get

$$[H_{n-2}, [H_{n-2}, N]] = \begin{bmatrix} 0 & 0 & 0 & 4a & 2(b+b') \\ 0 & 0 & x^t & 2(b+b') & 4a \\ 0 & -x & 0 & y & 0 \\ -4a & -2(b+b') & -y^t & 0 & 0 \\ -2(b+b') & -4a & 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix must be Toeplitz, we conclude $x = 0$, $y = 0$ and $b + b' = 0$. In other words, N has the following form:

$$N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a & b \\ 0 & 0 & 0 & \cdots & 0 & -b & a \\ 0 & 0 & & & & 0 & 0 \\ \vdots & \vdots & & N_2 & & \vdots & \vdots \\ 0 & 0 & & & & 0 & 0 \\ -a & b & 0 & \cdots & 0 & 0 & 0 \\ -b & -a & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

When $p = n - 3$, we partition N as follows:

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & -b & a \\ 0 & 0 & 0 & x^t & c & 0 & 0 \\ 0 & 0 & -x & N_3 & y & 0 & 0 \\ 0 & 0 & -c & -y^t & 0 & 0 & 0 \\ -a & b & 0 & 0 & 0 & 0 & 0 \\ -b & -a & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We get

$$[H_{n-3}, [H_{n-3}, N]] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2a & 2(b+c) \\ 0 & 0 & 0 & 0 & 2a & -4b & 2a \\ 0 & 0 & 0 & x^t & 2(b+c) & 2a & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & -2a & -2(b+c) & -y^t & 0 & 0 & 0 \\ -2a & 4b & -2a & 0 & 0 & 0 & 0 \\ -2(b+c) & -2a & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix is Toeplitz, we conclude that $x = 0$, $y = 0$, $b + c = 0$, $a = 0$ and $b = 0$. In other words, N has the following form:

$$N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & N_3 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Case 1. $n \geq 7$. Then $n - 3 > \frac{n}{2}$. By the above argument, the last three rows and columns are zero. Hence, by Lemma 3.3, $N = 0$.

Case 2. $n = 6$. Then $n - 2 > \frac{n}{2}$. As above using test matrices $H_{n-1} = H_5$ and $H_{n-2} = H_4$ we see that N has the following form:

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & -b & a \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & -c & 0 & 0 & 0 \\ -a & b & 0 & 0 & 0 & 0 \\ -b & -a & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[H_3, [H_3, N]] = \begin{bmatrix} 0 & 0 & 0 & 0 & 2a & 2(b+c) \\ 0 & 0 & 0 & 2a & -4b & 2a \\ 0 & 0 & 0 & 2(b+c) & 2a & 0 \\ 0 & -2a & -2(b+c) & 0 & 0 & 0 \\ -2a & 4b & -2a & 0 & 0 & 0 \\ -2(b+c) & -2a & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix must be Toeplitz we conclude that $b + c = 0$, $a = 0$ and $b = 0$. Thus $N = 0$.

Case 3. $n = 5$. Then $n - 2 > \frac{n}{2}$. As above using $H_{n-1} = H_4$ and $H_{n-2} = H_3$, we see that N has the following form:

$$N = \begin{bmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & -b & a \\ 0 & 0 & 0 & 0 & 0 \\ -a & b & 0 & 0 & 0 \\ -b & -a & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$[H_1, [H_1, N]] = \begin{bmatrix} 0 & a & 3b & a & * \\ -a & 0 & -a & 6b & a \\ -3b & a & 0 & -a & 3b \\ -a & 6b & a & 0 & a \\ * & -a & -3b & -a & 0 \end{bmatrix}.$$

Since this matrix is Toeplitz we conclude $a = b = 0$ and hence $N = 0$.

Case 4. $n = 4$. Then $n - 1 > \frac{n}{2}$. As above using $H_{n-1} = H_3$, we see that N has the following form:

$$N = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & b' & a \\ -a & -b' & 0 & 0 \\ -b & -a & 0 & 0 \end{bmatrix}$$

and

$$[H_2, [H_2, N]] = \begin{bmatrix} 0 & 0 & 4a & 2(b + b') \\ 0 & 0 & 2(b + b') & 4a \\ -4a & -2(b + b') & 0 & 0 \\ -2(b + b') & -4a & 0 & 0 \end{bmatrix}.$$

Since this matrix must be Toeplitz we conclude that $b + b' = 0$. In other words, N has the following form:

$$N = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & a \\ -a & b & 0 & 0 \\ -b & -a & 0 & 0 \end{bmatrix}.$$

Then we have

$$[H_1, [H_1, N]] = \begin{bmatrix} 0 & 4b & a & * \\ -4b & 0 & -8b & a \\ -a & 8b & 0 & 4b \\ * & -a & -4b & 0 \end{bmatrix}.$$

Since this matrix must be Toeplitz we conclude that $b = 0$. Now, we use

$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ as a test matrix. Then

$$[\mathbf{i}K_1, [\mathbf{i}K_1, N]] = -a\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}.$$

Since this matrix is Toeplitz we conclude that $a = 0$ and hence $N = 0$.

Case 5. $n = 3$. Then $n - 1 > \frac{n}{2}$. As above using $H_{n-1} = H_2$ we see that N has the following form:

$$N = \begin{bmatrix} 0 & a & b \\ -a & 0 & a \\ -b & -a & 0 \end{bmatrix}.$$

Let

$$H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \text{ and } K_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then H_1, $\mathrm{i}K_1$, $\mathrm{i}K_2$, $H_1\mathrm{i}K_1$ and $H_1 + \mathrm{i}K_2$ are all Hermitian Toeplitz. Hence

$$[H_1 + \mathrm{i}K_1, [H_1 + \mathrm{i}K_1, N]] - [H_1, [H_1, N]] - [\mathrm{i}K_1, [\mathrm{i}K_1, N]]$$

$$= [H_1, [\mathrm{i}K_1, N]] + [\mathrm{i}K_1, [H_1, N]]$$

$$= \mathrm{i}b[H_1, [K_1, K_2]] + \mathrm{i}a[K_1, [H_1, K_1]] + \mathrm{i}b[K_1, [H_1, H_2]]$$

$$= \mathrm{i}a[K_1, [H_1, K_1]] + \mathrm{i}b([H_1, [K_1, K_2]] + [K_1, [H_1, K_2]])$$

$$= -2\mathrm{i}a \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} + \mathrm{i}b \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

This matrix must be Toeplitz and that b must be zero. In other words, N has the form aK_1. We also consider

$$[H_1 + \mathrm{i}K_2, [H_1 + \mathrm{i}K_2, N]] - [H_1, [H_1, N]] - [\mathrm{i}K_2, [\mathrm{i}K_2, N]]$$

$$= [H_1, [\mathrm{i}K_2, N]] + [\mathrm{i}K_2, [H_1, N]]$$

$$= \mathrm{i}a([H_1, [K_2, K_1]] - [K_2, [H_1, K_1]])$$

$$= \mathrm{i}a \begin{bmatrix} 2 & 0 & 6 \\ 0 & -4 & 0 \\ 6 & 0 & 2 \end{bmatrix}.$$

This matrix must be Toeplitz and that a must be zero. Hence $N = 0$.

Theorem 3.5. *Let M be an $n \times n$ Hermitian matrix, where $n \geq 3$. Then M satisfies the Toeplitz preserving property (T) iff M is a real scalar matrix.*

Proof. $(\Leftarrow)$ Let $M = aI$, where a is a real number. Then $[A, M] = 0$ for every matrix A. Of course, M has the Toeplitz preserving property.

$(\Rightarrow)$ Let $Re\,M$ and $Im\,M$ be the real and imaginary parts of M; in symbols, $(Re\,M)(j, k) := Re(M(j, k))$ and $(Im\,M)(j, k) := Im(M(j, k))$. In particular, $Re\,M$ and $Im\,M$ are $n \times n$ real matrices with $M = Re\,M + \mathbf{i} Im\,M$. Since M is Hermitian, $Re(M)$ is real symmetric and $Im(M)$ is real skew-symmetric. Since M satisfies the property (T), $Re(M)$ satisfies the property (TR) and $Im(M)$ satisfies the property (TS). Now, by Theorem 3.2, $Re(M) = aI + bJ$. And, by Theorem 3.3, $Im(M) = 0$. It remains to show that $b = 0$. Suppose that $b \neq 0$, then $[\mathbf{i}K_1, [\mathbf{i}K_1, J]]$ is Toeplitz. Note that $Z^t = JZJ$ and

$$\begin{aligned}[K, J] &= [Z - JZJ, J] \\ &= ((Z - JZJ)J - J(Z - JZJ)) \\ &= 2(ZJ - JZ) = 2[Z, J].\end{aligned}$$

Hence we have

$$\begin{aligned}[\mathbf{i}K_1, [\mathbf{i}K_1, J]] &= -[K_1, [K_1, J]] = -2[Z - JZJ, [Z, J]] \\ &= (-2)((Z - JZJ)(ZJ - JZ) - (ZJ - JZ)(Z - JZJ)) \\ &= (-2)((Z^2J - ZJZ - JZJZJ + JZJJZ) \\ &\quad - (ZJZ - ZJJZJ - JZ^2 + JZJZJ)) \\ &= (-2)(Z^2J - ZJZ - JZJZJ + JZ^2 - ZJZ \\ &\quad + Z^2J + JZ^2 - JZJZJ)\end{aligned}$$

$$= (-2)(2Z^2J - 2ZJZ - 2JZJZJ + 2JZ^2)$$

$$= (-4)(Z^2J + JZ^2 - ZZ^tJ - JZZ^t).$$

This matrix is not Toeplitz; for example, when $n = 3$, we have

$$Z^2J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$JZ^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$ZZ^tJ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$JZZ^t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and hence

$$Z^2J + JZ^2 - ZZ^tJ - JZZ^t = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We conclude that $b = 0$. In other words, $Re(M)$ is a real scalar and so is M.

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